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Weak locale quotient morphisms and locally connected frames

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Abstract

Weak locale quotient morphisms between frames are defined, and it is proved that local connectedness of frames is preserved under weak locale quotient morphisms, without any form of choice principle. This provides affirmative answers to problems posed by Baboolal and Banaschewski (1991) and by Chen (1992).

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Recently, Baboolal and Banaschewski [1] posed the following problem: to prove, without the Boolean Ultrafilter Theorem, that any regular subframe of a locally connected compact regular frame is locally connected. As remarked by Chen [2], this is a special case of a more general problem: to prove that local connectedness is inherited by closed subframes (i.e. closed locale quotients).

Chen solved the original problem in [2], but left the more general problem open. In this paper we shall show that the second problem also has an affirmative solution. We shall define a class of frame homomorphisms which we call 'weak locale quotient morphisms', and which include open and closed frame injections as special cases, and we shall prove that local connectedness is preserved by such morphisms. This generalizes the classical result [7] that local connectedness is preserved under quotient maps of spaces.

In this paper, we shall work mainly in the 'algebraic' setting of the category **Frm** of frames and frame homomorphisms. Readers who prefer to think in terms of locales may make the necessary translations for themselves. For a general background on frames and locales, we refer to [3].

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For a frame A, we let $N(A)$ denote the frame of all nuclei on A, ordered pointwise, and we identify its order-theoretical dual $N^\bullet(A)$ with the poset of all sublocales of A under inclusion: that is, $N^\bullet(A) = \{A_j \mid j \in N(A)\}$, where A_j is the image of j for each $j \in N(A)$. Given a frame homomorphism $f: A \to B$, we let $f^-: N^\bullet(A) \to N^\bullet(B)$ denote categorical inverse image (i.e. pushout along f), and $f[\ ]: N^\bullet(B) \to N^\bullet(A)$ denote categorical direct image: that is, for a sublocale B_l of B, $f[B_l]$ is the image of the composite homomorphism $lf: A \to B \to B_l$. (The nucleus corresponding to $f[B_l]$ is just $f_* l_* lf$, where f_* and l_* denote the right adjoints of f and l respectively.) Then $f[\ ]$ is left adjoint to f^- [5], so $f[\ ]$ preserves joins of sublocales. Similarly, f^- preserves meets; but it also preserves finite joins [3, II 2.8], and so in particular it preserves complemented elements.

Definition 1. A frame homomorphism $f: A \to B$ is called a *weak locale quotient morphism* if it is injective and any sublocale A_j of A such that A_j is complemented in $N^\bullet(A)$ and $f^-(A_j)$ is an open sublocale of B is an open sublocale of A.

Since open sublocales are the complements of closed sublocales, we easily obtain

Lemma 2. *An injection $f: A \to B$ in* **Frm** *is a weak locale quotient morphism if and only if each sublocale A_j of A such that A_j is complemented in $N^\bullet(A)$ and $f^-(A_j)$ is closed in B is a closed sublocale of A.*

Proposition 3. *Equalizers in* **Frm** *are weak locale quotient morphisms.*

Proof. Let $f: A \to B$ be an equalizer of $s, t: B \rightrightarrows C$ in **Frm**, and let A_j be a complemented sublocale of A such that $f^-(A_j)$ is an open sublocale $B_{u(b)}$ of B. Since $sf = tf$, we have

$$C_{u(s(b))} = s^-(B_{u(b)}) = t^-(B_{u(b)}) = C_{u(t(b))}$$

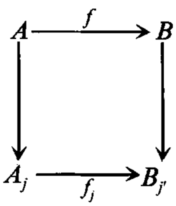
and hence $s(b) = t(b)$, i.e. $b = f(a)$ for some $a \in A$. Thus we have an open sublocale $A_{u(a)}$ of A such that $f^-(A_{u(a)}) = f^-(A_j)$. But f is injective, and from the description of $N(A)$ as the splitting frame of A [4] it follows that f^- is injective on complemented sublocales; so $A_j = A_{u(a)}$. Hence f is a weak locale quotient morphism. $\square$

Remark 4. Since quotient maps of spaces (in the usual sense) are coequalizers in the category of spaces, they give rise to equalizers in **Frm** and hence to weak locale quotient maps by the above proposition. However, we do not know whether the converse of the proposition is true: this is the reason why we included the adjective 'weak' in Definition 1.

We next show that open and closed monomorphisms in **Frm** are weak locale quotient morphisms. We recall that a frame homomorphism $h: A \to B$ is said to be *closed* if $h_*(h(a) \vee b) = a \vee h_*(b)$ for any $a \in A$, $b \in B$. This condition is equivalent to saying that $h(a) \le b \vee h(c)$ implies $a \le h_*(b) \vee c$ for all $a, c \in A$ and all $b \in B$; so h is closed if and only if $h[\ ]: N^\bullet(B) \to N^\bullet(A)$ maps closed sublocales to closed sublocales, as in [5]. h is said to be *open* if it has a left adjoint (equivalently, preserves joins) and preserves the Heyting implication: as shown in [4], this is equivalent to the condition that $h[\ ]: N^\bullet(B) \to N^\bullet(A)$ maps open sublocales to open sublocales.

We require the following lemma obtained in [6]:

Lemma 5. *Let $f: A \to B$ be a monomorphism in* **Frm**, *and A_j a complemented sublocale of A. Then in the pushout square*



f_j is also a monomorphism.

Proposition 6. *Open and closed monomorphisms in* **Frm** *are weak locale quotient morphisms.*

Proof. Let $f: A \to B$ be a frame monomorphism, A_j a complemented sublocale of A. By Lemma 5 and the uniqueness of image factorizations in **Frm**, we have $A_j = f[f^-(A_j)]$. So the result for open monomorphisms follows immediately from the characterization of open morphisms given above; that for closed monomorphisms follows from Lemma 2 and the similar characterization of closed morphisms. $\square$

Next, we recall the definitions of connectedness and local connectedness.

Definition 7. Let A be a frame. An element $a \in A$ is called *connected* if $a \neq 0$ and, whenever we have $b, c \in A$ with $b \wedge c = 0$ and $b \vee c = a$, then either $b = 0$ or $c = 0$. We say A is a *connected frame* if its top element 1 is connected. And we say A is *locally connected* if every element of A can be expressed as a join of connected elements.

It is readily verified that an element $a \in A$ is connected if and only if the corresponding open sublocale $A_{u(a)}$ is a connected frame.

Definition 8. A sublocale A_j of A is called a *connected component* of A if it is connected and, given any connected sublocale A_l with $A_j \cap A_l \neq \{1\}$, we have $A_l \subseteq A_j$. (Here $\{1\}$ denotes the smallest element of $N^\bullet(A)$.)

Let $C(A)$ denote the set of all connected sublocales of A. We define an equivalence relation $\sim$ on $C(A)$ as follows: $A_j \sim A_k$ if and only if there exists a finite sequence of sublocales, beginning with A_j and ending with A_k, such that the intersection of any two consecutive members of the sequence is nontrivial. It is easy to see that the join of any equivalence class in $C(A)$ is a connected frame, and hence that it is a connected component of A. (However, A need not be the join of its connected components; indeed, $C(A)$ may be empty.)

Lemma 9. *Any connected component of a frame is a closed sublocale.*

Proof. In [1, Lemma 1.6], it is shown that the closure of a connected sublocale is connected. Hence if A_j is a connected component of A, its closure $\overline{A_j}$ is also connected; but since $A_j \subseteq \overline{A_j}$ we must have $A_j = \overline{A_j}$. □

Lemma 10. *Let $\{A_{u(s)} \mid s \in S\}$ be a family of open sublocales of A, and A_j an arbitrary sublocale of A. Then*

$$A_j \cap \bigvee_{s \in S} A_{u(s)} = \bigvee_{s \in S} (A_j \cap A_{u(s)}).$$

Proof. From [4, Proposition 1, p. 28], we have $j \vee u(s) = s \to j(-)$ for every $s \in S$. Thus we have

$$\bigvee_{s \in S} (j \vee u(s)) = \bigwedge_{s \in S} (s \to j(-)) = \left(\bigvee S\right) \to j(-)$$

$$= j \vee u\left(\bigvee S\right) = j \ \vee \bigwedge_{s \in S} u(s)$$

in $N(A)$. Translating this across the anti-isomorphism from $N(A)$ to $N^\bullet(A)$ yields the result. □

Lemma 11. *A frame A is locally connected if and only if, for every open sublocale $A_{u(a)}$ of A, the connected components of $A_{u(a)}$ are also open and $A_{u(a)}$ is their join.*

Proof. The sufficiency of the condition is obvious. Conversely, let A be locally connected; let $A_{u(a)}$ be an open sublocale of A and A_j one of its connected components.

By assumption, we can write a as a join $\bigvee S$ of connected elements of A; so we have

$$A_j = A_j \cap A_{u(a)} = A_j \cap \bigvee_{s \in S} A_{u(s)} = \bigvee_{s \in S} (A_j \cap A_{u(s)})$$

by Lemma 10. But for every $s \in S$ we have either $A_j \cap A_{u(s)} = \{1\}$ or $A_j \cap A_{u(s)} = A_{u(s)}$, since A_j is a connected component of $A_{u(a)}$. Hence we have expressed A_j as a join of (connected) open sublocales of A, and so in particular it is open.

Finally, since $A_{u(a)}$ is a join of connected (open) sublocales and every connected sublocale is contained in a connected component, $A_{u(a)}$ is also the join of its connected components. □

Lemma 12. *If $f: A \to B$ is a monomorphism in* **Frm** *and B can be represented as a join of connected sublocales, then A can also be represented as a join of connected sublocales.*

Proof. Suppose $B = \bigvee_j B_j$, where each B_j is connected. Then $f[B] = \bigvee_j f[B_j]$ since $f[\ ]$ preserves joins of sublocales, and the $f[B_j]$ are also connected since a subframe of a connected frame is connected. But $f[B] = A$ since f is a monomorphism. □

Theorem 13. *If $f: A \to B$ is a weak locale quotient morphism and B is locally connected, then A is also locally connected.*

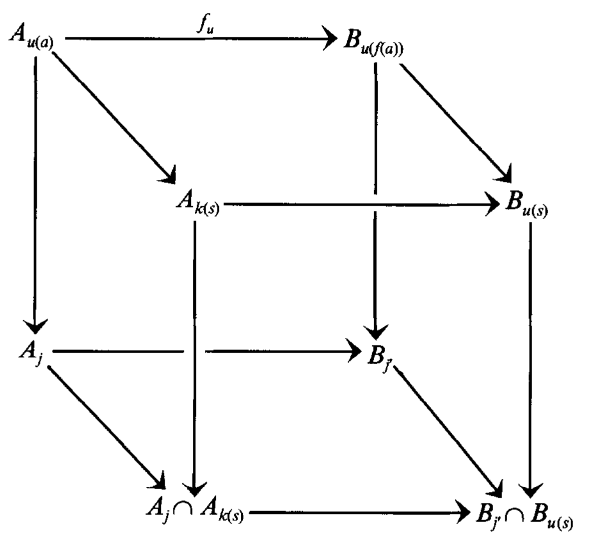
Proof. Let $A_{u(a)}$ be an open sublocale of A. If we form the pushout

$$\begin{array}{ccc} A & \xrightarrow{\ f\ } & B \\ \big\downarrow & & \big\downarrow \\ A_{u(a)} & \xrightarrow[\ f_u\ ]{} & B_{u(f(a))} \end{array}$$

then f_u is a monomorphism by Lemma 5. Since B is locally connected, $B_{u(f(a))}$ can be expressed as a join of connected sublocales; so by Lemma 12 $A_{u(a)}$ can be expressed as a join of connected sublocales, and hence in particular as the join of its connected components.

It remains to show that the connected components of $A_{u(a)}$ are open. Let A_j be one such component. By Lemma 9, A_j is closed in $A_{u(a)}$ and hence complemented (in $N^{\bullet}(A_{u(a)})$, and hence also in $N^{\bullet}(A)$). So it suffices to show that $f^{-}(A_j) = B_{j'}$ say is an open sublocale of B. Let $B_{u(f(a))} = \bigvee_{s \in S} B_{u(s)}$ where each $B_{u(s)}$ is connected; then by

Lemma 10 we have $B_{j'} = \bigvee_{s\in S} B_{u(s)}(B_{j'} \cap B_{u(s)})$. Suppose $B_{j'} \cap B_{u(s)} \neq \{1\}$ for some $s \in S$. Then, writing $A_{k(s)}$ for $f[B_{u(s)}]$, we have a diagram



where the front bottom arrow exists because the left face is a pushout. Since $B_{j'} \cap B_{u(s)} \neq \{1\}$, we conclude that $A_j \cap A_{k(s)} \neq \{1\}$. Since A_j is a component of $A_{u(a)}$ and $A_{k(s)}$ (as a subframe of $B_{u(s)}$) is connected, this forces $A_{k(s)} \subseteq A_j$. But $f[\ ]$ is left adjoint to f^-, so this is equivalent to $B_{u(s)} \subseteq B_{j'}$. Thus $B_{j'}$ is the join of those $B_{u(s)}$ for which $B_{j'} \cap B_{u(s)} \neq \{1\}$; in particular, it is an open sublocale of B, as required. □

Combining the theorem with Proposition 6, we obtain

Corollary 14. *Let $f: A \to B$ be a frame monomorphism which is either open or closed. If B is a locally connected frame, then so is A.*

If A is regular and B is compact, then any frame morphism $A \to B$ is closed; thus we also obtain the result originally conjectured by Baboolal and Banaschewski [1]:

Corollary 15. *A regular subframe of a compact (regular) locally connected frame is locally connected.*

Finally, combining the theorem with Proposition 3, we also have

Corollary 16. *If $f: A \to B$ is an equalizer in* **Frm** *and B is locally connected, then so is A.*

By Remark 4, Corollary 16 includes the corresponding result for spaces as a special case.

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